Math 275D Lecture 5 Notes

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1 Brownian Motion Filtrations and Markov Property

1.1 Two filtrations for Brownian motion

Last time, we introduced the σ-fields

$$\mathcal{F}_s^+ := \bigcap_{t>s} \mathcal{F}_t^0, \qquad \mathcal{F}_s^0 := \sigma(B(s'), s' \leq s).$$

We have that $\mathcal{F}_s^0 \subseteq \mathcal{F}_s^+$, and these are not the same because we have an event in $\mathcal{F}_s^+ \setminus \mathcal{F}_s^0$:

$$\left\{ \limsup_{t \to s^+} \frac{B(t) - B(s)}{t - s} \geq \frac{1}{2} \right\}.$$

Later, we will show that the sets in $\mathcal{F}_s^+ \setminus \mathcal{F}_s^0$ differ from sets in $\mathcal{F}_s^0$ by null sets. In applications, we can use both of these σ-fields interchangeably. Usually, we use $\mathcal{F}_s^+$ because $\mathcal{F}_s^+ = \lim_{t \to s^+} \mathcal{F}_t^+ = \lim_{t \to s^+} \mathcal{F}_t^0$. That is, we want a right continuous filtration.

How do we show that $\mathcal{F}_s^+$ is almost the same as $\mathcal{F}_s^0$? For any bounded random variable Z on Ω, we can define the conditional expectations

$$\mathbb{E}[Z \mid \mathcal{F}_s^+], \qquad \mathbb{E}[Z \mid \mathcal{F}_s^0].$$

We will show that these are equal a.s. for any such Z.

Here is an application of this result (which is very difficult to prove without it).

Example 1.1. Fix s. What is

$$A_s = \mathbb{P}(\inf\{t > s : B(t) > B(s)\} = s\}?$$

This should be the same as

$$A_0 = \mathbb{P}(\inf\{t > 0 : B(t) > 0\} = 0).$$

Naively, you may assume that the answer should be 1/2; but in fact, it is 1.

Proposition 1.1. $\mathbb{P}(\inf\{t > 0 : B(t) > 0\} = 0) = 1$.

Proof. Let B_0 be the event $\inf_{B(t)>0}\{t : t > 0\} = 0$, so $A_0 = \mathbb{P}(B_0)$. Then B_0 is $\mathcal{F}_0^+$-measurable. Then

$$\mathbb{E}[\mathbb{1}_{B_0} \mid \mathcal{F}_0^0] = \mathbb{E}[\mathbb{1}_{B_0} \mid \mathcal{F}_0^+] = \mathbb{1}_{B_0}.$$

But $\mathcal{F}_0^0 = \{\varnothing, \Omega\}$, since $B(0) = 0$. So $\mathbb{E}[\mathbb{1}_{B_0} \mid \mathcal{F}_0^0] = \mathbb{E}[\mathbb{1}_{B_0}]$. This gives us

$$\mathbb{1}_{B_0} = \mathbb{E}[\mathbb{1}_{B_0}].$$

So $\mathbb{P}(B_0) = 1$ or 0.

To show that $\mathbb{P}(B_0) \neq 0$, let C_0 be the event $\inf\{t > 0 : B(t) < 0\}$. Then $\mathbb{P}(C_0) = \mathbb{P}(B_0)$, and $\mathbb{P}(B_0 \cup C_0) = 1$. So $\mathbb{P}(B_0) > 0$. □

1.2 Markov property of Brownian motion

Now let's prove this crucial result about $\mathcal{F}_s^+$ and $\mathcal{F}_s^0$. Here is some notation.

Let $Y : C_B(\mathbb{R}) \to [-M, M]$ for some $M > 0$. Then define $\mathbb{E}_x[Y] := Y(B(\cdot) + x)$ for $x \in \mathbb{R}$; this says that we input a Brownian motion with $B(0) = x$. Similarly, we define $\mathbb{E}_x[Y \mid \mathcal{F}] := \mathbb{E}[Y(B(\cdot) + x) \mid \mathcal{F}]$.

Recall the **shift operator** $\theta_s : C(\mathbb{R}) \to C(\mathbb{R})$ given by $\theta_s(f)(x) = f(x + s)$.

Lemma 1.1 (Markov property of Brownian motion)**.** *With the same notation as above,*

$$\mathbb{E}_x[Y \circ \theta_s \mid \mathcal{F}_s^+] = \mathbb{E}_{B(s)}[Y].$$

Remark 1.1. The right hand side is the expectation of $Y(\tilde{B}(\cdot))$, where $\tilde{B}$ is a Brownian motion independent of B such that $\tilde{B}(0) = B(s)$. So the right hand side is $\mathcal{F}_s^0$-measurable, while the left hand side is $\mathcal{F}_s^+$-measurable. The difficulty of proving this statement comes from this aspect.

Remark 1.2. Recall the similarity to the Markov property for Markov chains. There is a strong version of this property akin to the strong Markov property.